SYNTHESIS OF SINEWAVE OSCILLATOR BASED ON THE MODIFIED VAN DER POL EQUATION USING MELNIKOV THEORY

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The paper presents a method for synthesis of a sinewave oscillator governed by the modified Van der Pol equation. The synthesis is based on the Melnikov theory, whereupon the generated sinusoidal oscillations have in advance assigned amplitude and frequency. A Simulink model of the oscillator is proposed and investigated.

Keywords: Limit cycles, Melnikov function, sinewave oscillator

1. INTRODUCTION

The most general approach to synthesis of oscillators consists of finding a differential equation satisfying preliminarily assigned properties and then modeling this differential equation by electronic circuit. To this effect the present paper gives a method for synthesis of sinewave oscillator, whereupon the generated sinusoidal oscillations have in advance assigned amplitude and frequency. Under the given amplitude and frequency of the sinusoidal oscillations and with the aid of the Melnikov theory, we obtain the modified Van der Pol equation. Then this equation is modeled by Simulink and an appropriate electronic circuit.

2. BASIC NOTIONS OF THE MELNIKOV THEORY

Consider the following differential equation

$$\ddot{x} - \varepsilon(1 - cx^2)\dot{x} + \omega_0^2 x = 0 . \qquad (1)$$

where x is a physical quantity (real function of the time t), ω_0 and c are constants, ε is a small parameter, i.e. $|\varepsilon/\omega_0| << 1$, and $(\overset{\bullet}{\ }) \equiv d/dt$.

When $c = 1$ equation (1) becomes Van der Pol equation. Further we shall refer to Eq. (1) as a modified Van der Pol equation.

Equation (1) can be represented by the following autonomous system of differential equations

$$\left| \begin{array}{l} \dot{x} = \omega_0 y \\ \dot{y} = -\omega_0 x + \varepsilon(1 - cx^2) y \end{array} \right. . \qquad (2)$$

The function $g(x,y)=(1-cx^2)y$ describe the perturbation in the system (2) and Eq. (1). According to the Melnikov theory, the perturbation functions play a central role with the arising of limit cycles, or self-sustained oscillations. It is well-known that the dynamical systems governed by the Van der Pol equation (with $c=1$) generate steady-state sinusoidal oscillations having amplitude 2. The Melnikov theory provides a mean to find such a perturbation that the system is to generate sinusoidal oscillations having in advance assigned amplitude. Generally this is the idea for synthesis a sinewave oscillator, which is exposed in the present paper.

Putting $\varepsilon = 0$ in system (2) we obtain the unperturbed system

$$\left|\begin{array}{l} \dot{x} = \omega_0 y \\ \dot{y} = -\omega_0 x \end{array}\right., \qquad (3)$$

which is Hamiltonian with Hamiltonian function

$$H(x,y) = \frac{\omega_0}{2}x^2 + \frac{\omega_0}{2}y^2. \qquad (4)$$

The solution of the system (3) in the phase plane represents a one-parameter family of closed orbits obtained by the constant level sets of Hamiltonian function, i.e.

$$\Gamma_0(h): H(x,y) = \frac{\omega_0}{2}x^2 + \frac{\omega_0}{2}y^2 = h, \qquad h \in (0,\infty). \qquad (5)$$

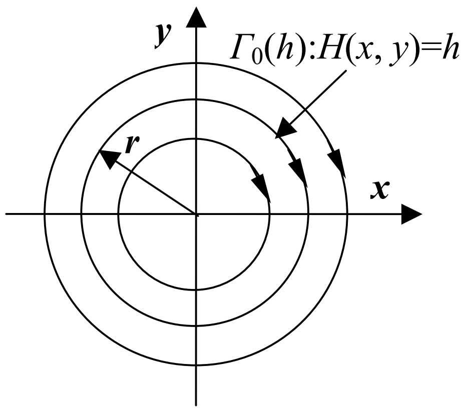

Fig.1. Phase portrait of the unperturbed system

The phase trajectory $\Gamma_0(h)$ is a circle (fig. 1) with a centre (0, 0) and a radius

$$r = \sqrt{\frac{2h}{\omega_0}}. \qquad (6)$$

The solution of system (3) in the time domain is given by sinusoidal functions whose amplitude A_m is equal to the radius r of the corresponding phase trajectory, i.e.

$$x = \varphi_0(t) = A_m \sin\omega_0 t = r\sin\omega_0 t = \sqrt{\frac{2h}{\omega_0}}\sin\omega_0 t\,. \tag{7a}$$

$$y = \psi_0(t) = A_m \cos\omega_0 t = r\cos\omega_0 t = \sqrt{\frac{2h}{\omega_0}}\cos\omega_0 t\,. \tag{7b}$$

The period of the solution $(x, y) = (\varphi_0(t), \psi_0(t))$ with respect to the time t is $T_0 = 2\pi/\omega_0$.

Consider the Poincare section Σ along the positive $y-$axis and assume that Σ is parameterized by the Hamiltonian levels, i.e. by the values of $h \in (0, \infty)$. Under these circumstances the Melnikov function for the system (2) defined on Σ has the following forms [1], [2]

$$M(h) = \oint_{\Gamma_0(h)} g\,dy = \int_0^{T_0} g(\varphi_0, \psi_0)\dot{\psi}_0\,dt\,. \tag{8}$$

Replacing the perturbation $g(x, y)$ and after simple calculations we obtain the expressions for the Melnikov function and its derivative [3], [4]

$$M(h) = \frac{2\pi}{\omega_0}h\left(1 - \frac{c}{2\omega_0}h\right), \qquad M'(h) = \frac{2\pi}{\omega_0}\left(1 - \frac{c}{\omega_0}h\right). \tag{9}$$

According to the Melnikov theory, if there exists a value $h = h_0 > 0$ for which the following conditions hold

$$M(h_0) = 0\,, \qquad M'(h_0) \neq 0\,, \tag{10}$$

then for a sufficiently small $\varepsilon \neq 0$ there exists h_ε in an $O(\varepsilon)$ neighborhood of h_0, such that in the system (2) arises a limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ The limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ is localized in an $O(\varepsilon)$ neighborhood of the curve $\Gamma_0(h_0) : H(x, y) = h_0$ and tends to $\Gamma_0(h_0)$ as $\varepsilon \to 0$. When $\varepsilon M'(h_0) < 0$ the limit cycle is stable and when $\varepsilon M'(h_0) > 0$ – unstable.

It follows from Eq. (9) that the Melnikov function has a single nonzero root $h_0 = 2\omega_0/c$. The derivative at h_0 is $M'(h_0) = -2\pi/\omega_0 < 0$. The last inequality implies that the limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ arising around the Hamiltonian level h_0 is stable at $\varepsilon > 0$ and unstable at $\varepsilon < 0$.

In practice the limit cycle $\Gamma_\varepsilon(h_\varepsilon)$ coincides with the circle $\Gamma_0(h_0)$. Hamiltonian levels are connected with the radii of the corresponding phase trajectories by Eq.(6). Replacing the root $h_0 = 2\omega_0/c$ in Eq. (6) we can oblain an expression for the amplitude of the generated sinusoidal oscillations

$$A_m = r_0 = \sqrt{\frac{2h_0}{\omega_0}} = \sqrt{\frac{4}{c}}\,. \tag{11}$$

In this case the final expression for the generated steady-state sinusoidal oscillations is

$$x(t) = A_m \sin\omega_0 t = r_0 \sin\omega_0 t = \sqrt{\frac{4}{c}} \sin\omega_0 t\,. \tag{12}$$

When $c = 1$ we get $A_m = 2$, which has to be expected, since in this case we have Van der Pol equation.

Having these results at disposal it is possible to synthesize oscillator systems that are based on the modified Van der Pol equation.

3. Simulink Modeling and Synthesis of Modified Van der Pol Equation

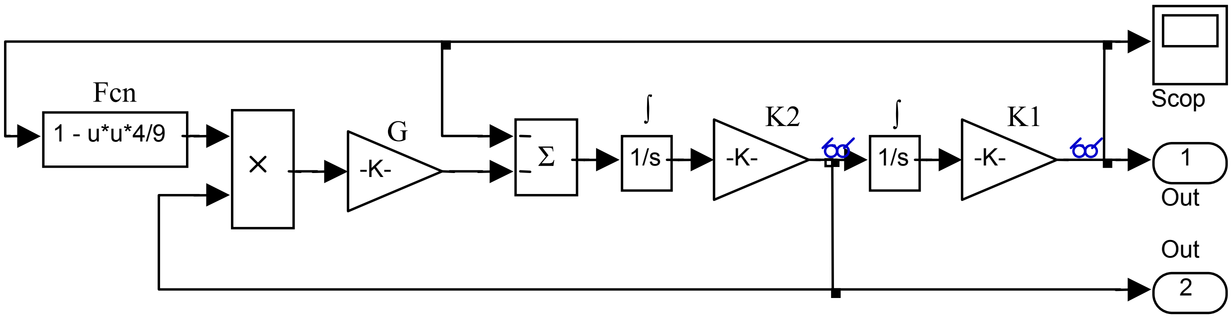

Fig.2. Simulink model of the modified Van der Pol equation

Fig. 2 shows the Simulink model of modified Van der Pol equation. It is easy to establish that for this model the following equation is valid

$$\frac{d^2x}{dt^2} - Gk_2(1 - cx^2)\frac{dx}{dt} + k_1k_2x = 0\,. \tag{13}$$

By introducing the quantities

$$\varepsilon = Gk_2\,,\ \omega_0 = \sqrt{k_1k_2}\,, \tag{14}$$

the last equation becomes identical with Eq. (1).

Now we can formulate the procedure for synthesis of a modified Van der Pol oscillator system, having oscillations with in advance assigned amplitude A_m and frequency ω_0. The procedure is reduced to determining the coefficients in Eq.(1) (or Eq.(13)) and includes the following step:

1) Determining the quantity c according to Eq.(11), that is

$$c = \frac{4}{A_m^2}\,; \tag{15}$$

2) Determining the quantities ε and ω_0 according to Eq.(14). It is necessary to be fulfilled the inequality $|\varepsilon/\omega_0| << 1$.

As an application of the proposed procedure we shall consider an example of synthesis of oscillator system.

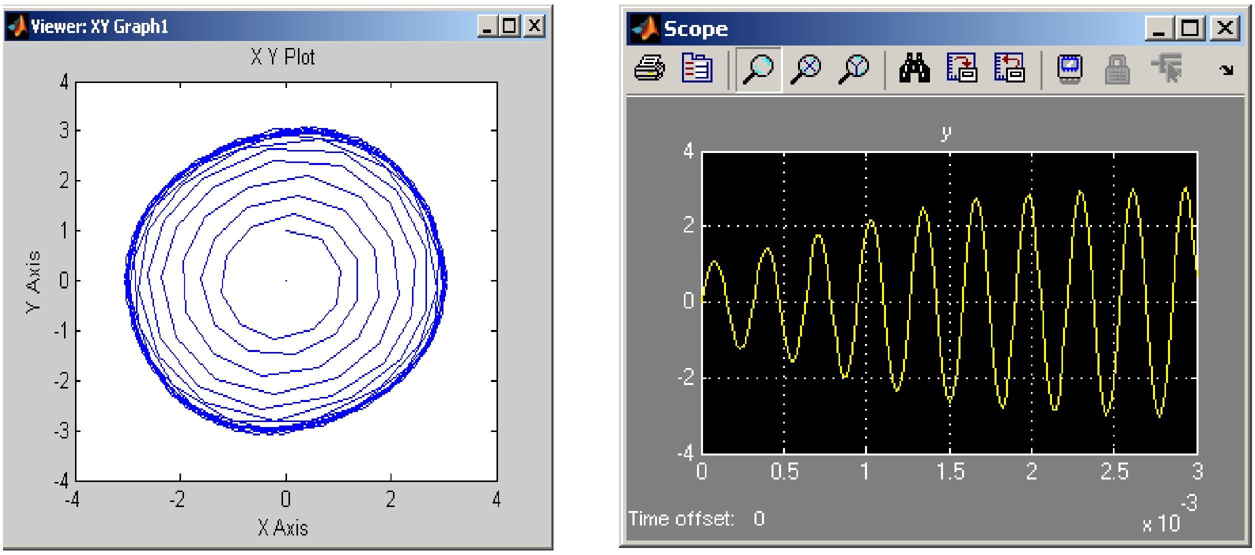

Fig.3. Phase portrait and oscillations obtained by Simulink modeling of the modified Van der Pol equation

Example : Find an oscillator system equation having steady-state sinusoidal oscillations with an amplitude $A_m = 3$ V and a frequency $f = 3183$ Hz ($\omega_0 = 20000$ rad/s).

Solution : According to the above mentioned procedure we obtain $c = 4/9$; $k = k_1 = k_2 = \omega_0 = 20000$; $G = 0{,}11$. In this way Eq(1) (or Eq.(13)) is completely determined. The results obtained by Simulink modeling are shown in Fig. 3.

4. CONLUSIONS

The paper presents a method for synthesis of sinewave oscillator under in advance assigned amplitude and frequency. The synthesis is based on the Melnikov theory applied to modified Van der Pol equation. The analytical results coincide completely with the results obtained by Simulink modeling.

5. REFERENCES

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