THE LOSSLESS COMPRESSION BY POLYPHASE FILTERING

Jaroslav Vrána

Miloslav Filka

Department of Telecommunications, FEEC, BUT, Purkyňova 118, 612 00 Brno, Czech Republic, phone: +420 54114 9217, e-mail: xvrana02@stud.feec.vutbr.cz

In this paper a dual channel mirrored filter bank and condition for their perfect reconstruction are described. The filter bank may be used in wavelet transform computing. Description of wavelet transform computing algorithm named lifting follow. Lifting algorithm is faster method of wavelet transform computing. The basis of this algorithm is in replacing all filters by their polyphase equivalents. The polyphase equivalents are transformed to individual lifting steps. A condition, which the individual lifting steps must fulfill, are obtained by analysis of this structure. The conditions are computed for one dual lifting and one primal lifting and two dual lifting and one primal lifting. The wavelets are simply computed from the conditions. If those wavelets are used in filter bank then the filter bank is filter bank with perfect reconstruction. A wavelet can be designed from these conditions and it can be adapted and used for lossless data compression.

Keywords: wavelet transform, compression, lifting

1. INTRODUCING

The wavelet transform is used e.g. in image compression or enhance signal in noise enhancing. This is relative new method which is usable for signal processing. Several methods for wavelet transform computing exist, with various computing complexity and wavelet type. Computing using convolution, FFT or lifting algorithm is some of the possible way. First the way from convolution method to lifting method is briefly mathematically described. When the lifting algorithm is used for wavelet transform with long wavelet, its computing complexity is one half in comparison with convolution method. The next part describes the design of wavelet from individual lifting steps so that it fulfills a condition for filter banks with perfect reconstruction.

2. WAVELET TRANSFORM COMPUTING

The basic block scheme for computing wavelet transform using filter banks is in Fig. 1. The input signal is decomposed into two bands by high pass filter $H_0(z)$ and low pass filter $H_1(z)$, [3]. Their bandwidths need not to be generally equal. The reconstruction of original signal from individual subbands is realized by two filters, low pass filter $G_1(z)$ and high pass filter $G_0(z)$, and subsequent summation of their outputs. In the case of filter bank with perfect reconstruction the output signal is identical with input signal.

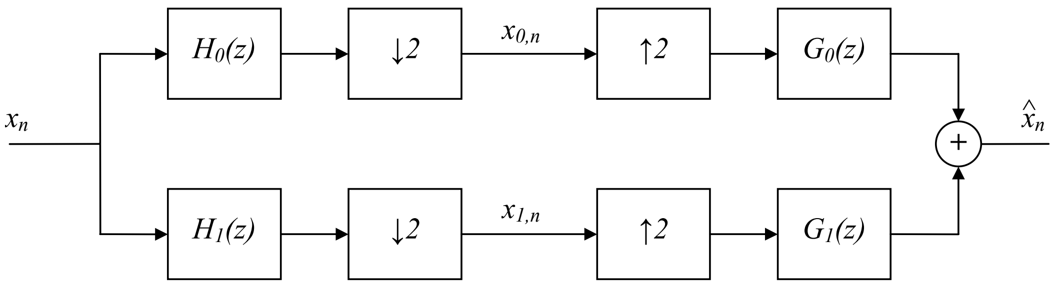

Fig. 1: *The block scheme for wavelet transform computing*

The following equations must be fulfilled in the case of filter bank with perfect reconstruction.

$$H_0(z)\cdot G_0(z)+H_1(z)\cdot G_1(z)=2z^{-k} \tag{2}$$

$$H_0(-z)\cdot G_0(z)+H_1(-z)\cdot G_1(z)=0, \tag{3}$$

where the left hand side of equation (2) correspond to sum of transfer functions at individual branches and the right hand side of equation expresses the delay originated by the filter bank. The equation (3) expresses the condition for aliasing removing which was originated by downsampling of subbands signals. The equation (3) may be expressed by following equations

$$H_0(z)=G_1(-z) \tag{4}$$

$$H_1(z)=-G_0(-z). \tag{5}$$

The low pass and high pass filters (see Fig. 1) need not to be of high order and their frequency characteristic can be overlapped. The transfer function of the cascade of the filters $H_0(z)$ and $G_0(z)$ (H_1 and G_1) on the frequency *fs/2* has to be 0,5. When these cascades are mirrored filters, then the following condition must be fulfilled.

$$P_0(z)=P_1(-z), \tag{6}$$

where $P_0(z)$ is transfer function of cascade $H_0(z)$ and $G_0(z)$ and $P_1(z)$ is transfer function of cascade $H_1(z)$ and $G_1(z)$. The individual filters in decomposition or reconstruction filter bank need not necessary to be mirrored. In detail the problematics of dual channel filter banks is described in [3].

3. LIFTING ALGORITHM

The basis of lifting algorithm is in substitution of filters from Fig. 1 by their polyphase equivalent and subsequent transformation to lifting steps [1], [2]. Every filter is replaced by two filters separately for even and odd signal samples. Now the input samples are split into even and odd samples. Even samples are directly filtered by two filters for even samples, while odd samples are first delayed and subsequently they are filtered like the even samples. The resulting sequences (decomposed by wavelet decomposition) are just sum of two output filters. So a cross structure originate. During a wavelet reconstruction a four filters are used in cross structure again. For correct interleaving of even and odd output samples it is necessary either delay even samples by one sample or accelerated odd samples by one sample. In this case the delay is chosen because of their feasibility.

The passage of a signal through this system can be simple described by matrix operations. The system can by described by the following equation (while spliter and merger block are omitted)

$$\begin{pmatrix} z\hat{X}_s(z) \\ \hat{X}_l(z) \end{pmatrix} = \begin{pmatrix} G_{0,s}(z) & G_{1,s}(z) \\ G_{0,l}(z) & G_{1,l}(z) \end{pmatrix} \begin{pmatrix} H_{0,s}(z) & H_{0,l}(z) \\ H_{1,s}(z) & H_{1,l}(z) \end{pmatrix} \begin{pmatrix} X_s(z) \\ z^{-1}X_l(z) \end{pmatrix} \tag{7}$$

For system with perfect reconstruction, the cascade of two foursomes of filters has unit transfer function. The system needs not to have zero delay, but it can have any but constant delay. Condition for perfect reconstruction is described by equation (8), where z^{-k} means constant delay and **I** means unit matrix.

$$\begin{pmatrix} G_{0,s}(z) & G_{1,s}(z) \\ G_{0,l}(z) & G_{1,l}(z) \end{pmatrix} \begin{pmatrix} H_{0,s}(z) & H_{0,l}(z) \\ H_{1,s}(z) & H_{1,l}(z) \end{pmatrix} = z^{-k}\mathbf{I}\,. \tag{8}$$

May $P_0(z)$ is the matrix containing $H(z)$ elements and $P_1(z)$ is the matrix containing elements $G(z)$. Then the equation (8) can be written in the form

$$\mathbf{P_1}(z)\mathbf{P_0}(z) = z^{-k}\mathbf{I}\,. \tag{9}$$

The matrix $P_0(z)$ and $P_1(z)$ can be resolved to so called primal a dual lifting, [1], [2]. A novel method called lifting originates in merging both above mentioned approaches. If the transfer functions can not be resolved to primal and dual lifting matrixes product, more steps in cascade ca follow one after another as long as an matrix with just nonzero constant elements in main diagonal the remains from the original one. In this case the primal and dual lifting are alternating. One pair of primal and dual lifting is called one lifting step. The connection of m lifting steps is mathematically formulated by using matrix

$$\mathbf{P_0}(z) = z^{-j}\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \cdot \prod_{i=1}^{m} \left\{ \begin{pmatrix} 1 & 0 \\ S_i(z) & 1 \end{pmatrix} \begin{pmatrix} 1 & T_i(z) \\ 0 & 1 \end{pmatrix} \right\}. \tag{10}$$

A block diagram depicting the above mathematically described situation is shown in Fig. 2.

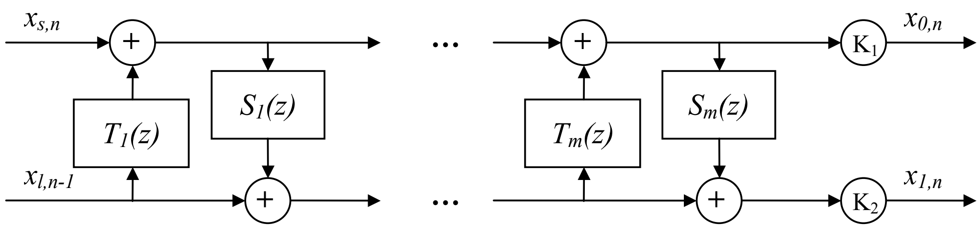

Fig. 2: *Block scheme of multistep lifting in decomposition*

The wavelet reconstruction can be analogically converted to the computation with usage of multistep lifting algorithm computing. The block scheme of multistep lifting wavelet reconstruction is mirrored like wavelet decomposition. All summation elements are replaced by difference.

4. General Wavelet Design

The wavelet design is based on lifting algorithm. The block scheme is described by equation (10). The conditions (2) a (3) must be fulfilled to the input signal be able decomposed and perfect reconstructed by wavelet transform. The equation (3) is fulfilled by choice reconstructions filters according to equations (4) a (5). For check condition (2) the transfer functions are expressed by their polyphase equivalents (11).

$$H(z) = H_s(z) + z^{-1}H_l(z) \tag{11}$$

After substitution equation (4), (5) and (11) into equation (2), the result equation is

$$H_{o,s}(z)H_{1,l}(z) - H_{1,s}(z)H_{0,l}(z) = 2z^{-k+1} \tag{12}$$

At comparison this equation with matrix $P_0(z)$, this equation is computation of its determinant. The determinant is equal to delay of passage through filter banks. May one lifting step (product of matrix primal and dual lifting) is $L_i(z)$.

$$\mathbf{L_i}(z) = \begin{pmatrix} 1 & 0 \\ S_i(z) & 1 \end{pmatrix} \begin{pmatrix} 1 & T_i(z) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & T_i(z) \\ S_i(z) & S_i(z)T_i(z)+1 \end{pmatrix} \tag{13}$$

If number of lifting steps increase by one, the transfer matrix is followed.

$$\mathbf{L_{i+1}}(z)=\begin{pmatrix} l_{i;1,1}+l_{i;2,1}T_{i+1}(z) & l_{i;1,2}+l_{i;2,2}T_{i+1}(z) \\ l_{i;1,1}S_{i+1}(z)+l_{i;2,1}[S_{i+1}(z)T_{i+1}(z)+1] & l_{i;1,2}S_{i+1}(z)+l_{i;2,2}[S_{i+1}(z)T_{i+1}(z)+1] \end{pmatrix} \quad (14)$$

After solving this equation, it can be seen that the addition of one lifting step not changes determinant. The determinant of transfer matrix of any number of lifting steps only depends on determinant of outputs multipliers by constants.

$$\det\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}=2z^{-k+j+1} \quad (15)$$

Because multiplication by constants K_1 and K_2 not originates delay, the result equation is

$$K_1 \cdot K_2 = 2\,. \quad (16)$$

It can be seen from this result that the designed filter bank can be perfectly reconstruct for any transfer function $T_i(z)$ and $S_i(z)$. Number of lifting steps is not important. The next conditions for wavelets imply low pass and high pass filters transfer function.

$$|H_0(1)|=0; |H_0(-1)|=\sqrt{2} \quad (17)$$

$$|H_1(1)|=\sqrt{2}; |H_1(-1)|=0\,. \quad (18)$$

The transfer function may be expressed in polyphase form (11). Solution of this equation system is followed equation (19).

$$\mathbf{P_0}(1)=\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (19)$$

The individual coefficients can be simply computed by equation (10) and (19). In the case one lifting step is used, these equations have followed solution.

$$K_1=\frac{1}{\sqrt{2}};\; K_2=\sqrt{2} \quad (20, 21)$$

$$T(1)=-1;\; S(1)=\frac{1}{2}\,. \quad (22, 23)$$

The transfer functions $T(z)$ and $S(z)$ written as a polynomial form results in final conditions for individual coefficients.

$$\sum_{i=0}^{N-1} t_i = -1; \ \sum_{j=0}^{M-1} s_j = \frac{1}{2}, \qquad (24, 25)$$

where N is number of $T(z)$ polynomial coefficients and M is number of $S(z)$ polynomial coefficients. In the case two dual lifting steps and one primal lifting step, the individual transfer functions must fulfill followed conditions.

$$K_2 = \frac{1}{2K_1}; \ T_1(1) = 1\frac{1}{K_1} \qquad (26, 27)$$

$$S_1(1) = K_1; \ T_2(1) = \frac{1-2K_1}{2K_1^2}. \qquad (28, 29)$$

Now more than one solution exists, because four equation and five variables are solved.

5. CONCLUSION

All of designed wavelets fulfill perfect reconstruction filter bank condition. All of designed wavelets have zero average value. The wavelet, computed from one step lifting, has all even coefficients equal zero expect first of them. This disadvantage is removed by adding one dual lifting step. The wavelets may be changed during wavelet transform computing. The wavelet can be adapted to actual data structure. This possibility of wavelet adaptation can be used in lossless data compression.

6. ACKNOWLEDGEMENTS

This work is supported by project FRVŠ 1534/2005/G1.

REFERENCES:

[1] DAUBECHIES, I., SWELDENS, W., *Factoring zavelet transforms into lifting steps*, September 1996, revise November 1997

[2] VALENS, C., *The Fast Lifting Wavelet Transform*, 1999

[3] VAIDYANATHAN, P. P.: *Multirate Systems and Filter Banks*, Prentice hall P T R, Englewood Cliffs, New Jersey, 1993, ISBN 0-13-605718-7

[4] JAN, J.: *Číslicová filtrace, analýza a restaurace signálů*, VUT Brno, 1997, ISBN 80-214-0816-2.

[5] SMÉKAL, Z., VÍCH, R.: *Číslicové filtry*, Academia, Praha, 2000, ISBN 80-200-0761-X.