# MAGNITUDE APPROXIMATION METHOD FOR VARIABLE IIR DIGITAL FILTERS DESIGN

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**Abstract:** A new approach to design variable IIR digital filters by using a cascade of $N$ equal individual filters of any order $n$ is proposed in this paper. First, the approximation method for lowpass filter specifications is outlined, then the general limitations of the method are investigated and a compact formula is derived. Next, the limitations for the main canonic approximations are investigated and convenient expressions are obtained. New first- and second-order filter sections, permitting very easy tuning of the cutoff frequency, are developed and the design and tuning strategies for highpass, bandpass and bandstop filters are proposed. Finally design examples are given and the superiority of the new method compared to other known method is demonstrated experimentally.

## 1. INTRODUCTION

Variable digital filters are important blocks and there are many design methods for FIR and IIR variable structures already known [1]. IIR variable filters are preferred for professional applications and they are usually designed by employing the allpass (frequency) transformations of Constantinides [2]. But when IIR prototypes are transformed, delay-free loops appear and no general design method, avoiding this problem, is known until now. There are some approximate methods to eliminate the delay-free loops, but, as a result, tuning is possible only over some limited frequency range. The best among all known is the method of Mitra, Neuvo and Roivainen (MNR)[3], employing truncated Taylor series expansions and based on parallel all-pass structures with real or complex coefficients. The main disadvantage of this method is the limited range of frequencies over which the LP/HP (lowpass/highpass) cutoff frequency and BP/BS (bandpass/bandstop) bandwidth may be tuned without degradation of its magnitude characteristics. A new approach, based on usage of equal first- or second-order sections and thus avoiding any approximate presentations was proposed in [4] and the range of tuning was considerably extended compared to that of the MNR filters. The design method in [4] is, however, quite not general and there are, actually, 4 different design procedures - one for approximation based on equal first-order terms and three for second-order terms. Our main aim in the present work is to try to develop a unified procedure, including not only the above mentioned four cases, but also approximation using equal terms of any order. Next, we will attempt to define the fundamental limitations of the method based on approximation with equal terms (sections). Finally, we shall try to develop some new first- and second-order structures, suitable for realization of this type of variable filters. We shall concentrate on obtaining and investigating mainly LP variable filters. Variable HP, BP and BS filters can easily be produced by applying a corresponding allpass transformation [1],[2] on the variable LP filter.

## 2. LOWPASS MAGNITUDE APPROXIMATION USING EQUAL INDIVIDUAL TRANSFER FUNCTIONS

### 2.1. Basic relations

The LP magnitude specifications to start the design of a variable filter are given in a standard way: passband (PB) from 0 to $\omega_p$ (for digital filters) or $\Omega_p$ (for analog), stopband (SB) from $\omega_s$ or $\Omega_s$ to infinity, maximum variation of the PB attenuation $A_p$ ,dB and minimum (SB) attenuation $A_s$,dB. In the process of approximation we are looking for a total transfer function $H(z)$ or $T(s)$ consisting of $N$ equal individual transfer functions $H_i(z)$ or $T_i(s)$, each of them of order $n$. Thus

$$H(z) = H_i^N(z); \qquad T(s) = T_i^N(s) \tag{1}$$

where

$$H_i(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots\ a_n z^{-n}}{1 + b_1 z^{-1} + b_2 z^{-2} + \dots\ b_n z^{-n}}, \tag{2}$$

$$T_i(s) = \frac{a_0 + a_1 s + a_2 s^2 + \dots\ a_m s^m}{b_0 + b_1 s + b_2 s^2 + \dots\ b_n s^n}, \quad m \le n. \tag{3}$$

In developing our approximation method we choose to work in analog domain and to use the well-known Characteristic function $k(\Omega)$. It has appeared, on one hand, to be extremely difficult to develop such a method directly in $z$-domain and was shown in [5], on the other hand, that this approach (approximation in s-domain with usage of $k(\Omega)$) must be preferred. The squared magnitudes of the individual and the total transfer functions will then be given as:

$$|T_i(\Omega)|^2 = 1/\left(1 + \varepsilon^2 |k(\Omega)|^2\right); \qquad |T(\Omega)|^2 = 1/\left(1 + \varepsilon^2 |k(\Omega)|^2\right)^N. \tag{4}$$

$k(\Omega)$ is taking always the squared values

$$0 \le |k(\Omega)|^2 \le k_p^2 \text{ in the PB}; \qquad k_s^2 \le |k(\Omega)|^2 \le \infty \text{ in the SB}, \tag{5}$$

the values of $k_p$ and $k_s$ depend on the selected approximation method, the PB and SB edge frequencies $\Omega_p$ and $\Omega_s$ and on the order $n$ and usually $k_p^2 << k_s^2$.

$|T_i(\Omega)|^2$ is varying within the limits:

$$t_p^2 \le |T_i(\Omega)|^2 \le 1 \text{ in the PB}; \quad 0 \le |T_i(\Omega)|^2 \le t_s^2 \text{ in the SB}, \tag{6}$$

$t_p$, and $t_s$ also depend on $\Omega_p$, $\Omega_s$, the approximation method and $n$, but, additionally, they are connected to the coefficient $\varepsilon$, predetermining the PB loss variations.

The total transfer function $|T(\Omega)|^2$ (4) will stay within the limits

$$T_p^2 \le |T(\Omega)|^2 \le 1 \text{ in the PB}; \qquad 0 \le |T(\Omega)|^2 \le T_s^2 \text{ in the SB}, \tag{7}$$

and it is obvious that

$$t_p^2 = 1/(1 + \varepsilon^2 k_p^2); \qquad t_s^2 = 1/(1 + \varepsilon^2 k_s^2);$$

$$T_p^2 = t_p^{2N},\ T_s^2 = t_s^{2N},\ A_p = -10\log T_p^2, dB,\ T_p^2 = 10^{-0.1A_p},\ T_s^2 = 10^{-0.1A_s}. \quad (8)$$

**2.2. Design procedure outline**

The proposed procedure is based on a gradual increase of $N$ and checking whether $T(\Omega)$ is entering within the given specifications. It includes the following steps:

1. Select the type of the individual transfer function $T_i(s)$ (could be any maximally flat of equiripple function) and its order $n$ and start with some initial value of $N$ (for example $N$=1). Thus, the values of $k_p^2$ and $k_s^2$ are readily known.
2. Calculate $t_p^2 = T_p^{2/N} = 10^{-A_p/(10N)}$. (9)
3. Compute the coefficient $\varepsilon^2$: $\varepsilon^2 = \left(1/k_p^2\right)\left[\left(1/t_p^2\right)-1\right]$. (10)
4. With this $\varepsilon^2$ and $T_s^2$ (8) check whether the following condition holds:

$$1/\left(1+\varepsilon^2 k_s^2\right)^N \le T_s^2. \quad (11)$$

If "*yes*", go to *step 5*. If "no" - increase $N$ by one and go back to *step 2*.

5. Determine the individual transfer function $T_i(s)$ using the values of $\varepsilon^2$ (10), $A_p/N$, $\Omega_p$, $\Omega_s$ and $n$. Many computer programs are available for this step.

**2.3. General limitations of the method**

It is clear that an approximation using $N$ equal terms is far from optimal. First, it is impossible to increase the SB attenuation without limits by increasing $N$, because it will also decrease $\varepsilon$. Then, having equal terms, we cannot place the zeros of the transfer function arbitrarily in the SB to ensure any given $A_s$. It is intuitively clear that such an approximation is good for narrowband filters with moderate requirements for the SB, but we need some more accurate expressions to evaluate the limitations in the process of design. Starting from $|T(\Omega)|^2$ (4) and substituting $\varepsilon^2$(10) and $t_p^2$(9), while taking the minimal value $k_s^2$ for $|k(\Omega)|^2$, we end at the following estimation for the maximal value of $|T(\Omega)|^2$ in the SB, predetermining the minimal SB attenuation:

$$T_{s\ \max}^2 = 1\Bigg/\left[1+\left(\frac{1}{T_p^{2/N}}-1\right)\frac{k_s^2}{k_p^2}\right]^N. \quad (12)$$

Investigating (12) for $N\to\infty$ we find that the minimal value of the SB attenuation will never exceed the limit

$$A_{s\max} = A_p k_s^2/k_p^2. \quad (13)$$

The limitation (13) is quite fundamental, it does not depend on the type of the filter (it is valid not only for LP filters), and as the reasonable values of $N$ (the number of the equal individual transfer functions used) are far from infinity, the limitations for the possible SB attenuation will be more severe than (13). The method, as expected, is not universal, and not every given specifications will be met.

It means that an additional step should be included in the design procedure from the previous section - once $k_p^2$ and $k_s^2$ are found (*step 1*), it must be checked whether $A_{s_{max}}$ (13) is higher then $A_s$ and if not, the specifications should be relaxed in order to make the design possible. As far as $H(e^{j\omega})$ can be given in the form (4), all relations obtained in sections 2.2 and 2.3 are valid also for approximations performed directly in $z$-domain.

**2.4. Limitations for the main classical approximations**

The evaluation of condition (13) can be simplified for each canonic approximation and, additionally, it can be transferred in $z$-domain. For this we can use the so called "rectangularity" - coefficient $r$, calculated in $s$- or $z$-domain:

$$r = \Omega_p / \Omega_s \quad \text{or} \quad r = \left(\tan\frac{\omega_p \tau}{2}\right) \Big/ \left(\tan\frac{\omega_s \tau}{2}\right), \tag{14}$$

where $\tau$ is the sampling interval.

1. *Butterworth type of individual transfer functions*

Taking into account the properties of the Butterworth approximation, we obtain the following limitation, corresponding to (13):

$$A_{s\max} = A_p / r^{2n}, dB. \tag{15}$$

For the case of equal first-order sections it gives the condition already derived in [4]

$$A_{s\max} = A_p \left(\tan^2\frac{\omega_p \tau}{2}\right) \Big/ \left(\tan^2\frac{\omega_s \tau}{2}\right), dB \tag{16}$$

and for equal second-order sections (without PB-ripples)

$$A_{s\max} = A_p / r^4, dB. \tag{17}$$

It is obvious that (16) is valid for all types of approximations.

2. *Chebishev type of individual transfer functions*

For this type of approximation we have derived the following general limitation formula:

$$A_{s\max} = A_p \operatorname{ch}^2\left(n \operatorname{Arch}\frac{1}{r}\right). \tag{18}$$

For second-order individual transfer functions it reduces to:

$$A_{s\max} = A_p\left[(2/r^2) - 1\right]^2. \tag{19}$$

3. *Elliptic type of individual transfer functions*

There is no an easy way to calculate $k_p^2$ and $k_s^2$ for this approximation without using elliptic functions. To get more compact results we normalize the analog domain frequency specifications by

$$\Omega_0 = \sqrt{\Omega_p \Omega_s}. \tag{20}$$

Then, using the theory of elliptic approximation [6], after some lengthy derivations, we obtain the following general limitations:

$$A_{s\max} = A_p \Big/ \left[ r^{2n} \prod_{i=1}^{n/2} \mathrm{sn}^8\left( \frac{(2i-1)K(r)}{n} \right) \right] \quad \text{for } n \text{ - even} \tag{21}$$

$$A_{s\max} = A_p \Big/ \left[ r^{2n+2} \prod_{i=1}^{\frac{n-1}{2}} \mathrm{sn}^8\left( \frac{(2i-1)K(r)}{n} \right) \right] \quad \text{for } n - \text{odd} \tag{22}$$

where $K(r)$ is a function of Jacobi, determined as an elliptic integral

$$K(r) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - r^2 \sin^2 \varphi}} = \int_0^1 \frac{dx}{\sqrt{(1-r^2)(1-r^2x^2)}} \tag{23}$$

and sn is a *sinus elliptic*-function.

Checking the realizability of given specifications throughout (21) - (23) could be a tremendous task for a practicing engineer. Assuming that in most practical cases $n$ will be $n = 2$ and using the relation

$$\mathrm{sn}(K(r)/2) = 1/\sqrt{1 + \sqrt{1 - r^2}}\,, \tag{24}$$

we finally obtain the following expression:

$$A_{s\max} = A_p \left(1 + \sqrt{1 - r^2}\right)^4 / r^4. \tag{25}$$

In trying to simplify expressions (22), (23) for any given $n$ in order to avoid the calculation of special functions, we succeeded to derive a very simple and compact formula for an approximate evaluation of the specifications:

$$A_{s\max} \approx \frac{A_p}{16} \exp\left[ n\pi^2 / \ln(16/(1 - r^2)) \right) \tag{26}$$

This formula should be used only for very selective filters having $0.9 \leq r < 1$. For lower values of $r$ it will produce an error of more than (20-30)% compared to the results from (21), (22).

## 3. LP FILTER SECTIONS IMPLEMENTATION

Each individual transfer function $H_i(z)$ of order $n$ can easily be realized by $n/2$ second-order sections plus one of first-order when $n$ is odd. These sections must meet the following requirements: (a) They must have a canonic number of multipliers in order to minimize the number of the tunable elements; (b) They must permit an independent tuning of the cutoff frequency $\omega_p$; It was shown in [4] that these sections should have low sensitivity for poles near $z$=1 (typical for narrowband LP filters).

The standard form of the first- and second-order transfer functions with unity DC gain, obtained after the approximation is

$$H_{LP1}(z)=\frac{1+g}{2}\frac{1+z^{-1}}{1+gz^{-1}};\qquad H_{LP2}(z)=\frac{1+g_1+g_2}{2+g_3}\frac{1+g_3z^{-1}+z^{-2}}{1+g_1z^{-1}+g_2z^{-2}}\quad(27)$$

where $g_3 = 2$ for the non-elliptic case.

It was shown in [4] that the best first-order section is the one given in Fig. 1a. For the second-order terms we propose to use the universal structure, shown in Fig. 1(b), which we have developed starting from the low-sensitivity section from Ref. [7]. The transfer functions at the LP output of Fig. 1a and at the Elliptic output of Fig. 1b are:

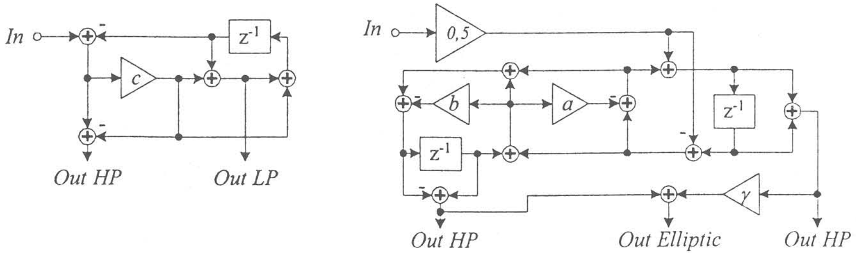



Fig.1. First (a) and second-order (b) low-sensitivity sections

$$H_{LP1}(z)=\frac{c(1+z^{-1})}{1+(1-2c)z^{-1}};$$

$$H_{Ellip}(z)=\frac{a(\gamma-1)+2-b}{2}\frac{1+2\frac{a(\gamma+1)-2+b}{a(\gamma-1)+2-b}z^{-1}+z^{-2}}{1+(-2+b+2a)z^{-1}+(1-b)z^{-2}}\quad(28)$$

The multiplier coefficients, obtained after sensitivity minimization, are calculated from those in (27) using the formulae:

$$c=0.5(1-g);\quad a=0.5(1+g_1+g_2);\quad b=1-g_2;\quad \gamma=\frac{-1+g_1-g_2}{1+g_1+g_2}\frac{g_3+2}{g_3-2}\quad(29)$$

The numerators of the transfer functions at the other outputs are

$$N_{HP1}(z)=1-z^{-1};\quad N_{LP2}=0.5(1+z^{-1})^2;\quad N_{HP2}=0.5(2-a-b)(1-z^{-1})^2\quad(30)$$

The cutoff frequency of the first-order section is easily tuned by trimming *c*. The second-order section of Fig. 1b is having the remarkable quality to have its cutoff frequency tuned only by changing the multiplier coefficient *a*. And, as $g_3$ also depends on *a*, the zeros of the transfer function will change correspondingly. It is important to mention, that 3 types of transfer function - namely LP and HP elliptic and BS as well - can be realized at the output given as *Out Elliptic* in Fig. 1b.

## 4. VARIABLE DIGITAL FILTERS DESIGN AND TUNING

Variable LP and HP filters, according to our design method, are obtained without traditional spectral transformations - tuning of the cutoff frequency is achieved by trimming of *c* or *a* ((28), Fig. 1). BP and BS variable filters with central frequency $\omega_0$ are obtained from the LP prototype after the transformation

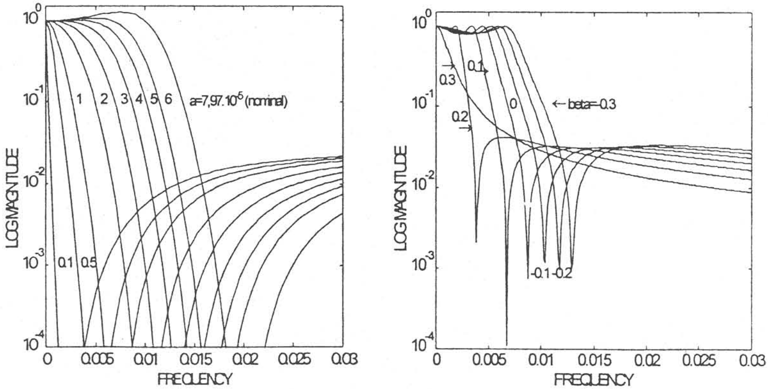
$z^{-1} = \mp z^{-1}\left(z^{-1} - \beta\right)/\left(1 - \beta z^{-1}\right)$ ("*minus*" for LP to BP transform); $\beta = \cos\omega_0$. (31)

If the cutoff frequency of the LP prototype is $\omega_{cp}$, the *BW* of the BP filter will be $BW_{BP} = \omega_{c2} - \omega_{c1} = \omega_{cp}$ and that of the BS filter - $BW_{BS} = 0.5\omega_{samp} - \omega_{cp}$, where $\omega_{samp}$ is the sampling frequency.

Given the BP or BS variable filters specifications, they must be converted to LP specification, then approximation using our method must be performed and finally the corresponding LP to BP or LP to BS transformation (31) has to be applied. The *BW* of the filter will be tuned by changing $\omega_{cp}$ and the central frequency - by trimming $\beta$. When second-order sections (Fig. 1b) are used, the approximation must be performed for the widest BW. If not, tuning in direction of widening of the BW may cause in some cases growing of the PB ripple and violation of the specification.

## 5. DESIGN EXAMPLE AND EXPERIMENTS

A variable LP filter is required with cutoff frequency tuned from $0.002\pi$ to $0.008\pi$ rad/s with specifications (for the central value $\omega_p = 0.005\pi$): $\omega_s = 0.001\pi$, $A_p = 2$dB and $A_s = 30$ dB. Approximation with equal second-order elliptic individual




(a) (b)

Fig.2. Tuning of LP filters designed following the proposed method (a) and MNR-method[3] (b)

transfer functions meets the specifications with *N*=4. The tuning of the realization based on the section of Fig.1b is illustrated in Fig.2a and it is seen that the entire range of cutoff frequencies from 0 to $0.001\pi$ is smoothly covered. The filter, designed according to the MNR-method [3] is of third-order and the results of tuning of the parallel allpass structures realizations are shown in Fig.2b. It is clear that the specifications for $A_s$ are violated far before reaching the requirements for the range of tuning of $\omega_p$.

## 6. CONCLUSIONS

A new very efficient approach to design variable IIR digital filters by using a cascade of $N$ equal individual filters of any order $n$ is developed in this paper. The method is very general and applicable for any order of the individual and the total transfer functions. Some of the formulae for evaluation of the limitations are so general, that they are valid for any type of approximation, any type of filters and even for both $s$- and $z$-domain. The new second-order section, proposed in the paper, is having the remarkable quality to have its cutoff frequency and the entire magnitude response tuned in very wide frequency range only by changing a single multiplier coefficient. The superiority of the new method compared to the famous MNR-method is demonstrated experimentally.

## REFERENCES


[1] G. Stoyanov and M. Kawamata, "Variable digital filters", *J. Signal Processing*, vol. 1, No.4, pp. 275-289, July 1997.

[2] A. G. Constantinides, "Spectral transformations for digital filters", *Proc. IEE*, vol. 117, pp. 1585-1590, Aug. 1970.

[3] S. K. Mitra, Y. Neuvo and H. Roivainen, "Design of recursive digital filters with variable characteristics", *Int. Journal Circuit Theory Appl.*, Vol. 18, pp. 107-119, 1990.

[4] G. Stoyanov, I. Uzunov and M. Kawamata, "Narrowband variable digital filters with independently tunable characteristics and minimum number of tunable elements", *Proc. ITC-CSCC'98*, Sokcho, Korea, vol. 1, pp. 613-616, July 13-15, 1998.

[5] A. Willson and H. Orchard, "Insights into digital filters made as the sum of two allpass functions", IEEE Trans. Circuits and Systems-I, vol. 42, No. 3, pp. 129-137, March 1995.

[6] A. Sedra and P. Brackett, Filter theory and design: Active and passive, Pitman Publ. Ltd., London, 1978.

[7] I. Topalov and G. Stoyanov, "A systematic approach to the design of low-sensitivity limit-cycle-free universal bilinear and biquadratic digital filter sections", Proc. 10'th European Conf. ECCTD'91, Copenhagen, Denmark, vol. 1, pp.213-222,. Sept. 2-6, 1991.