Limit cycles in the Van der Pol-Duffing equation

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Abstract – Limit cycles in the Van-der Pol-Duffing equation are investigated.There position and shape is determined by the zeros of Melnikov function. For the determination of Melnikov function the Van-der Pol-Duffing equation is presented as a perturbed Hamiltonian system. The Melnikov function is an integral over the periodic trajectories surrounding the centre of the unperturbed system. The periodic solution of the unperturbed system (Hamiltonian system) is presented by the elliptic function of Jacobi.

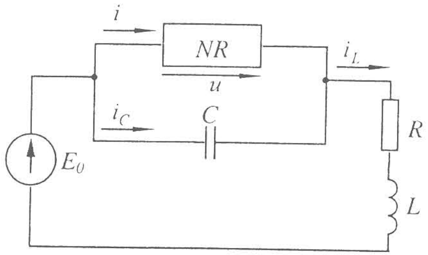

fig. 1

The electric circuit of an oscillator consisting of source of d.c. voltage $E_0 = u_0 = const$, resistance R, inductance L, capacitance C and nonlinear resistor NR, whose current-voltage characteristic has a negative resistance region has been presented on fig. 1.

It can be shown that the behaviour of such an oscillator is to be described by the Van der Pol-Duffing equation

$$\ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + ax + bx^3 = 0, \tag{1}$$

where $x = x(t)$ is the variable component of the normalized voltage u of the nonlinear resistor, while t is the normalized time. It has been accepted that $(\dot{\ }) = \frac{d}{dt}(\ )$. The constants a, b and ε in equation (1) depend on the values of the electromagnetic parameter in the circuit, whereupon $\varepsilon << 1$.

The subject of the current research is to determine and localize limit cycles in the Van der Pol-Duffing equation, when $a > 0$ and $b > 0$.

By mean of substitution $y = -\dot{x}$ equation (1) can be transformed into autonomous system

$$\begin{vmatrix} \dot{x} = -y = -y + f(x, y) \\ \dot{y} = ax + bx^3 + \varepsilon\left(1 - x^2\right)y = ax + bx^3 + g(x, y) \end{vmatrix}. \tag{2}$$

It can be regarded as one close to Hamiltonian system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = ax + bx^3 \end{cases}, \tag{3}$$

whose Hamiltonian being

$$H(x, y) = \frac{1}{2}y^2 + \frac{a}{2}x^2 + \frac{b}{4}x^4 = h, \qquad h \in (0, \infty) \tag{4}$$

where h is the parameter.

Functions

$$f(x, y) = 0 \qquad \text{and} \qquad g(x, y) = (1 - x^2)y, \tag{5}$$

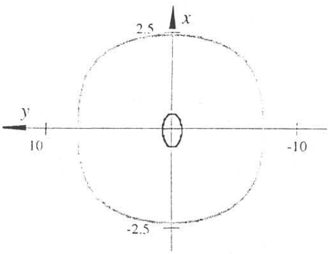

fig. 2

that take part in the equations of the perturbed system (2) represent perturbations of the nonperturbed system (3).

Phase portrait of the nonperturbed system (3) by definite values of a and b has been shown on fig. 2.

It is known that the determination and localization of limit cycles in the perturbed system (2) can be done by the zeros of Melnikov function (for the same system)

$$M(h) = \int_0^{T(h)} \left[f(\varphi_0, \psi_0)\dot{\psi}_0 - g(\varphi_0, \psi_0)\dot{\varphi}_0 \right] dt, \tag{6}$$

where $x = \varphi_0 = \varphi_0(t, h)$ and $y = \psi_0 = \psi_0(t, h)$ represent the periodic solution of the nonperturbed system (3) with a period $T = T(h)$

It can be proved that the periodic solution of the nonperturbed system (3) represents a couple of functions

$$\begin{cases} x = \dfrac{\sqrt{2}\sqrt{\dfrac{a}{b}}k\sqrt{1-k^2}}{\sqrt{1-2k^2}} sd(z,k) \\ y = -\dfrac{\sqrt{2}\sqrt{a}\sqrt{\dfrac{a}{b}}k\sqrt{1-k^2}}{1-2k^2} cd(z,k)\,nd(z,k) \end{cases} \qquad (7)$$

where $z = \dfrac{\sqrt{a}\,t}{\sqrt{1-2k^2}}$. The solution makes sense only when $0 \le k^2 \le 0{,}5$.

The elliptic functions of Jacobi [3]

$$sd = sd(z,k),\ cd = cd(z,k)\ \text{and}\ nd = nd(z,k), \qquad (7a)$$

that take part in the right sides of equation (7) are periodic concerning variable z and their module is k. Their periods concerning z are $4K$, $4K$ and $2K$ respectively, where $K = K(k)$ is the complete elliptic integral of first kind. The functions x and y period is $4K$.

It is easy to show that when

$$k^2 = \frac{1}{2} - \frac{a}{2\sqrt{a^2+4bh}}, \quad \text{resp.} \quad h = \frac{a^2k^2(1-k^2)}{b(1-2k^2)^2} \qquad (8)$$

functions x and y satisfy equation (4) as well.

By mean of equations (5), (7) and (7a) Melnikov function can be represented as follows

$$M(h) = -\int_0^{T(h)} (1-\varphi_0^2)\psi_0\dot{\varphi}_0 dt =$$
$$= -\frac{8a\sqrt{a}k^2(1-k^2)\sqrt{1-2k^2}}{b(1-2k^2)^2}\left[\frac{2ak^2(1-k^2)}{b(1-2k^2)}\int_0^K sd^2cd^2nd^2dz - \int_0^K cd^2nd^2dz\right] \qquad (9)$$

When calculating the integrals in the right side of (9) and the substitutions

$$\lambda = \frac{a}{b} > 0 \quad \text{and} \quad c = c(\lambda) = \frac{4\lambda+5}{2\lambda+10} \qquad (10)$$

are performed function $M(h)$ becomes

$$M(h)=\frac{8a\lambda\sqrt{1-2k^2}(2\lambda+10)}{15(1-2k^2)^3}\left[(1-k^2)(c-k^2)K-(2k^4-2k^2+c)E\right], \tag{11}$$

where $E=E(k)$ is the complete elliptic integral of second kind and k^2 is function of h.

The constant c in equations (10) and (11) satisfies the condition $0{,}5<c<2$, which results in $c-k^2>0$.

Finally equation (11) can be represented in the form

$$M(h)=\frac{8a\lambda\sqrt{1-2m}(2\lambda+10)(c-m)E}{15(1-2m)^3}[S(m)-T(m,c)], \tag{12}$$

where

$$m=m(h)=k^2=\frac{1}{2}-\frac{a}{2\sqrt{a^2+4bh}}, \tag{12a}$$

$$S(m)=\frac{K}{E}(1-m), \tag{12b}$$

and

$$T(m,c)=\frac{2m^2-2m+c}{c-m}. \tag{12c}$$

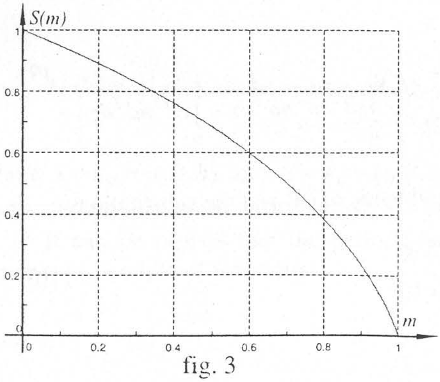

fig. 3

For function $S(m)$ in equation (12b) it holds the following : $S(0)=1$, $S(1)=0$, $S'_m(0^+)=-0{,}5$ and $S'_m(1^-)=-\infty$. In the interval $0\le m\le 1$ it is monotonously decreasing and concave down. Its graph has been shown on fig. 3.

For function $T(m,c)$ in equation (12c) the following holds : $T(0,c)=1$ and $T(0{,}5,c)=1$. In the interval $0\le m\le 0{,}5$ it has a single minimum $T_{\min} = \dfrac{2(2c-1)\left(c-\sqrt{c(c-0{,}5)}\right)}{\sqrt{c(c-0{,}5)}}$, which is obtained when $m = c - \sqrt{c(c-0{,}5)}$. Moreover $T'_m(0^+,c) = -\dfrac{1}{c} < -0{,}5$, so that $T'_m(0^+,c) < S'_m(0^+)$.

Due to this in the whole interval $0 \le m \le 0{,}5$ the graph of the function $T(m,c)$ crosses the graph of function $S(m)$ simply and only once in a point with absciss m_0.

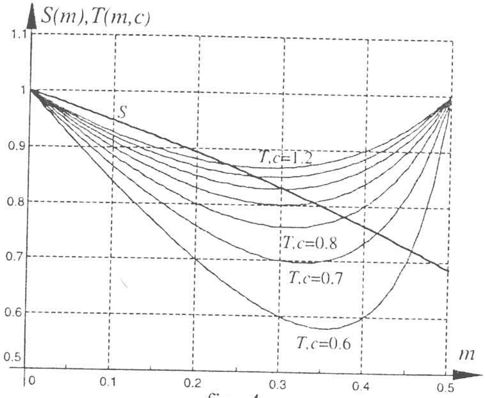

fig. 4

Under these conditions Melnikov function equals to zero when $m = m_0$, resp. $h = h_0$, where h_0 is determined from (8) when $k^2 = m_0$.

The graphs of functions $S(m)$ and $T(m,c)$ when $c = 0{,}6, 0{,}7, \ldots, 1{,}2$ have been shown on fig. 4.

It is to be seen that when $a > 0$ and $b > 0$ the perturbed system (2) has a single limit cycle located in the ε - neighbourhood of curve $\Gamma_0 : H(x,y) = h_0$.

Example: Determine and localize limit cycles of perturbed system (2) when $a = 5$ and $b = 4$.

The investigation is performed in the following order :

1) λ and c are calculated. Here

$$\lambda = \frac{a}{b} = \frac{5}{4} = 1{,}25 \qquad \text{and} \qquad c = \frac{4\lambda + 5}{2\lambda + 10} = \frac{4.1{,}25 + 5}{2.1{,}25 + 10} = 0{,}8$$

2) The graphs of functions $S(m)$ and $T(m,0{,}8)$ are built. The absciss m_0 of their crossing point is determined. Here $m_0 = 0{,}375$.

3) Parameter h_0 is calculated. Here

$$h_0 = \frac{a^2 m_0 (1 - m_0)}{b(1 - 2m_0)^2} = \frac{5^2.0{,}375.(1 - 0{,}375)}{4.(1 - 2.0{,}375)^2} = 23{,}4375.$$

Therefore when $a = 5$ and $b = 4$ the perturbed system (2) has a single limit cycle located in the ε-neighbourhood of curve $\Gamma_0 : 0{,}5.y^2 + 2{,}5.x^2 + x^4 = 23{,}4375$.

4) The period $T = T(h)$ of steady state oscillations is determined. Here

$$T = 4K\frac{\sqrt{1-2m_0}}{\sqrt{a}} = 4.1.7606\frac{\sqrt{1-2.(0.375)^2}}{\sqrt{5}} = 1.5747$$

5) The amplitude x_a of steady state oscillations is determined. Here

$$x_a = \frac{\sqrt{2}\sqrt{\frac{a}{b}}\sqrt{m_0}\sqrt{1-m_0}}{\sqrt{1-2m_0}}sd(K,k) = \frac{\sqrt{2}\sqrt{\frac{5}{4}}.\sqrt{0,375}\sqrt{1-0,375}}{\sqrt{1-2.0.375}}1,2649 = 1,9365$$

The phase portrait of system (2) has been shown on fig. 5. The solution of equation (1) in the area of variable t when $\varepsilon = 0,5$ has been shown on fig. 6.

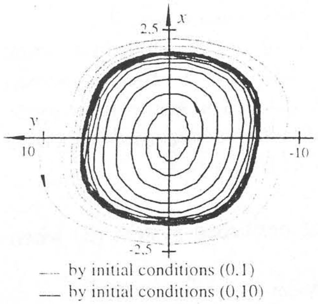

fig. 5

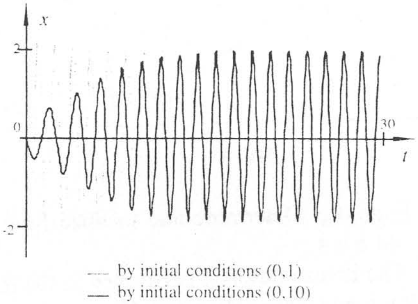

fig. 6

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