On the Representation of Planar Curves Using Wavelet Transform

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Abstract *— We present a procedure for representation of discrete-point parameterized coordinates of planar closed curve by an arbitrary numbee of samples. It consists in the approximation of the curve by piecewise constant functions followed by resampling, and can find an application in representation of planar curves using wavelets.*

I. INTRODUCTION

There is a higher number of different applications requiring the representation of planar closed curves such as pattern recognition, shape matching, computer graphics and image analysis. Many methods have been proposed to describe planar curves during the past three decades. Most recently Chuang and Jay Kuo [1] have considered multiscale planar curve descriptor by using the wavelet transform. This descriptor decomposes a curve into components of different scales, so that the coarsest components carry the global approximation information, while the finer scale components contain the local detailed information. Since the wavelet transform can be implemented via the cascade of quadrature mirror filter banks, the wavelet descriptor is computationally efficient and has a great potential for real-time applications.

The representation of closed planar curves by wavelet descriptor has many desired properties, including nonshrinking, invariance, uniqueness and stability, but also there are some constraints that have to be satisfied. The curves should be represented in terms of continuous parametric coordinate. Usually this is not a case and the planar curves are defined by discrete-point parameterized coordinates. Another constraint concerns curve's length: it should be divisible by N, where N is a power-of-two.

In this paper we present a procedure to overcome the foregoing constraints. It consists in the approximation of discrete-point coordinates by piecewise constant functions followed by resampling. At first, we briefly review the wavelet representation for planar curves [1] based on the theory of periodized wavelets [2].

II. PLANAR CURVE REPRESENTATION USING WAVELET TRANSFORM

The periodized wavelets are suitable for representation of closed curves. They can be expressed as a sum of copies of periodically continuous wavelets and constitute an orthonormal basis in the space $L^2([0,1])$.

The translations of the scaling function $\phi(t)$ for a certain $m \in \mathbf{Z}$

$$\phi_n^m(t) = 2^{-m/2}\phi(2^{-m}t - n), \quad n \in \mathbf{Z} \tag{1}$$

form an orthonormal basis for the wavelet subspaces V_m. $\{V_m\}_{m\in\mathbf{Z}}$ is a multiresolution approximation of the space $L^2(\mathbf{R})$. For each scaling function $\phi(t)$, one can determine the corresponding mother wavelet function $\psi(t)$ such that its dilations and translations

$$\psi_n^m(t) = 2^{-m/2}\psi(2^{-m}t - n), \quad m,n \in \mathbf{Z} \tag{2}$$

form an orthonormal basis for $L^2(\mathbf{R})$.

The periodic scaling and wavelet functions are defined as

$$\tilde{\phi}_n^m(t) = \sum_{l\in\mathbf{Z}} \phi_n^m(t-l), \quad \tilde{\psi}_n^m(t) = \sum_{l\in\mathbf{Z}} \psi_n^m(t-l). \tag{3}$$

The corresponding periodic multiresolution approximation spaces are

$$\tilde{V}^m = \overline{\mathrm{Span}\{\tilde{\phi}_n^m,\ n\in\mathbf{Z}\}}$$

and

$$\tilde{W}^m = \overline{\mathrm{Span}\{\psi_n^m,\ n\in\mathbf{Z}\}}. \tag{4}$$

with

$$\tilde{V}^{m-1} = \tilde{V}^m + \tilde{W}^m \tag{5}$$

as in the nonperiodic case. For other details concerning periodized wavelets we reffer to [2].

The finite-scale orthogonal wavelet expansion of $f(t) \in \mathrm{V}_{Mf}$ is:

$$f(t) = \sum_{n\in\mathbf{Z}_{M_c}}^{M_f} c_n^{M_f}\tilde{\phi}_n^{M_f}(t) = \sum_{n\in\mathbf{Z}_{M_c}}^{M_f} c_n^{M_c}\tilde{\phi}_n^{M_c}(t) + \sum_{m\in M_f}^{M_c}\sum_{n\in\mathbf{Z}_m} d_n^m \tilde{\psi}_n^m(t) \tag{6}$$

with

$$c_n^{M_c} = \int_0^1 f(t)\tilde{\phi}_n^{M_c}(t)dt \text{ and } d_n^m = \int_0^1 f(t)\tilde{\psi}_n^{M_c}(t)dt \tag{7}$$

It is important to notice that the finite-scale orthogonal wavelet transform can be computed using fast algorithm called forward and inverse discrete periodic wavelet transform (DPWT) [2].

A clockwise-oriented closed plane curve with parametric coordinates $x(t)$ and $y(t)$ is denoted by

$$\alpha(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad t(l) = l/L, \quad 0 \le l \le L \tag{8}$$

whith t normalized arc length, l arc length along the curve from a certain starting point t_0, and L total arc length.

The wavelet transform applied to the parameterized coordinates gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_a^M(t) \\ y_a^M(t) \end{bmatrix} + \sum_{m=M-m_0}^{M} \begin{bmatrix} x_d^m(t) \\ y_d^m(t) \end{bmatrix} \tag{9}$$

where

$$x_a^M(t) = \sum_n a_n^M \tilde{\phi}_n^M(t) \quad \text{and} \quad y_a^M(t) = \sum_n c_n^M \tilde{\phi}_n^M(t) \tag{10}$$

are the approximation coefficients at the scale M and

$$x_d^m(t) = \sum_n r_n^m \tilde{\psi}_n^m(t) \quad \text{and} \quad y_d^m(t) = \sum_n d_n^m \tilde{\psi}_n^m(t) \tag{11}$$

are the detailed signals at scale m. The different multiresolution representation of (9) can be obtained by

$$\begin{bmatrix} \hat{x}(k,t) \\ \hat{y}(k,t) \end{bmatrix} = \begin{bmatrix} x_a^M(t) \\ y_a^M(t) \end{bmatrix} + \sum_{m=k}^{M} \begin{bmatrix} x_d^m(t) \\ y_d^m(t) \end{bmatrix} \tag{12}$$

where $M - m_0 \leq k \leq M + 1$. The curves (12) provide a sequence of multiresolution approximations to the original curve.

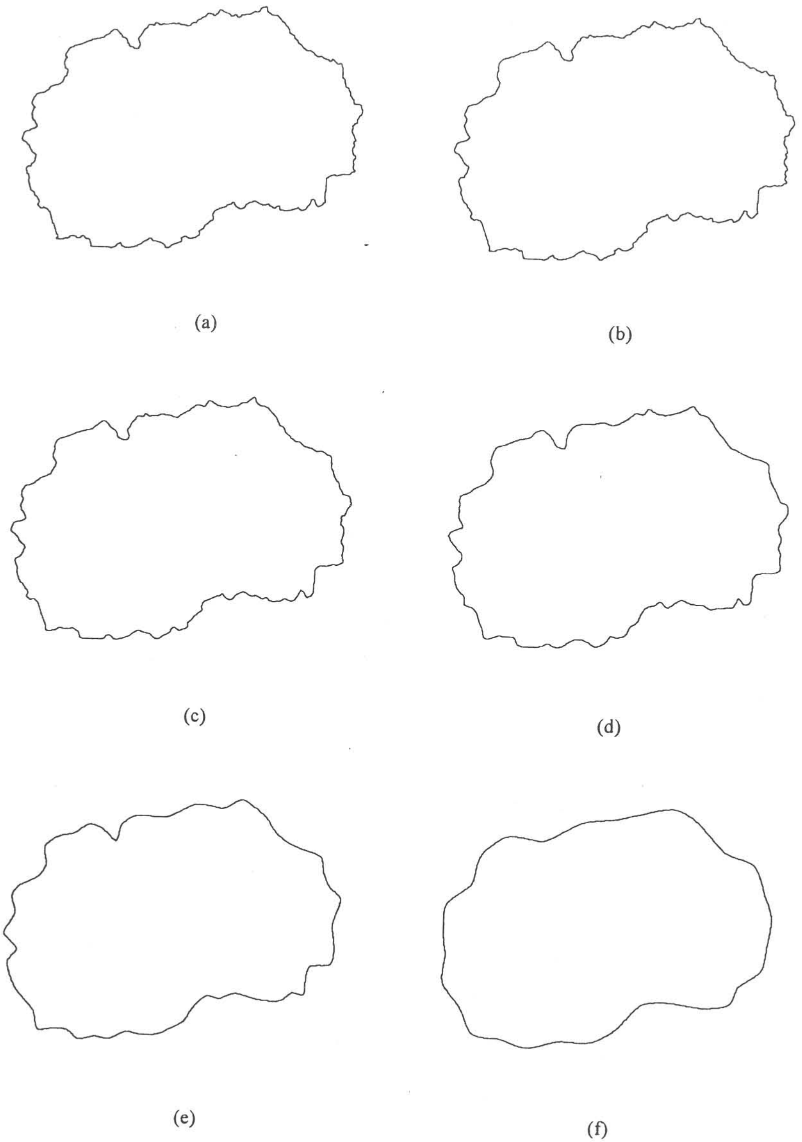
It is obvious that the representation of planar closed curves at very different approximation's scales requires longer sequences. In addition, it is desired that the sequences have a length N that is divisible by power-of-two. Generally this is not a case. In order to achieve an appropriate length the discrete-point parameterized curve can be firstly approximated to the continuous function and then adequately resampled. The simplest continuous approximation of the discretized curve is by piecewise constant function, as follows.

Let $x(n)$ and $y(n)$, $n = 0, 1, \cdots, N-1$ denote the discrete-point parameterized coordinates of a curve. The continuous approximation of the curve can be expressed as

$$\begin{aligned} \hat{x}(t) &= \sum_{i=0}^{N-1} x(i)[s(t - iT) - s(t - iT - T)] \\ \hat{y}(t) &= \sum_{i=0}^{N-1} y(i)[s(t - iT) - s(t - iT - T)] \end{aligned} \tag{13}$$

where $s(t)$ is the unit step function

$$s(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}. \tag{14}$$



(a) (b) (c) (d) (e) (f)

Fig. 1. Finite resolution approximations of the outline of Macedonia with Coiflet 6 wavelet base. (a) Original curve reprezented by 986 samples; (b) approximation scale $m=1$ with 512 nonzero wavelet coefficients; (c) $m=2$; (d) $m=3$; (e) $m=4$; (f) $m=5$ (32 coefficients).

The coordinates of the discrete approximation $x_d(i)$ and $y_d(i)$, $i = 0, 1, \cdots, N_d - 1$ are obtained by sampling $\hat{x}(t)$ and $\hat{y}(t)$

$$x_d(i) = \hat{x}(iNT/M),\ y_d(i) = \hat{y}(iNT/M),\quad i = 0, 1, \cdots, N_d - 1. \tag{15}$$

The length of sequences $x_d(i)$ and $y_d(i)$, N_d, is assumed to be a power-of-two.

As an example Fig. 1 shows a dyadic sequence of approximations of the outline of Macedonia. The wavelet base is Coiflet 6. The original curve (a) is represented by 968 points. The approximations (b), (c), (d), (e), and (f) are obtained by 512, 256, 128, 64, and 32 nonzero wavelet coefficients, respectively. All the computations are performed using the software package Wavelab [4].

III. Summary

We have considered a discrete-point parameterized planar closed curves and the sequences of their multiresolution approximations. The approximations at very different scales are acheived by approximating the discretized curves to the piecewise constant functions followed by appropriate resampling.

References

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